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Short Communication

A qualitative study of the solutions to the differential equation $\ddot{x} + (1 + \dot{x}^2)x = 0$

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The nonlinear oscillator differential equation [1]

$$\ddot{x} + (1 + \dot{x}^2)x = 0 \tag{1}$$

has the interesting feature that the application of the first-order harmonic balance method [2] gives the following functional relationship between the amplitude, A, and the angular frequency ω:

$$\omega(A) = \frac{2}{\sqrt{4 - A^2}}. \tag{2}$$

This result is obtained by use of the initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{3}$$

Inspection of Eq. (2) shows that $\omega(A)$ is not defined for amplitudes of magnitude equal to or larger than two in value. The issue that immediately arises is whether this restriction is an actual property of the solutions to Eq. (1) or is this an artifact of the harmonic balance method itself? It should also be indicated that even using more advanced techniques, such as the one constructed by Chatterjee [1], the angular frequency, $\omega(A)$, still has a singularity as a function of the amplitude; see for example Eq. (28) in Ref. [1].

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The purpose of this communication is to examine the properties of the solutions to Eq. (1) by studying the behavior of its trajectories in the two-dimensional phase space (x, y), where $y = \dot{x}$. The major advantage of doing this is that the general properties of the solutions can be determined without the application of any approximation method or explicit knowledge of the actual solutions to Eq. (1). The phase-space technique for two-dimensional dynamical systems is discussed in Refs. [2,3]. For completeness it should be stated that Eq. (2) is an example of a nonlinear oscillator having a velocity-dependent frequency [4].

To begin, Eq. (1) can be written as a system of two coupled, first-order differential equations

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = (1 + y^2)x. \tag{4}$$

The equilibrium solution or fixed-point for this system is located at $(\bar{x}, \bar{y}) = (0, 0)$. The first-order differential equation that must be solved to obtain the trajectories, $y = y(x)$, in the (x, y) phase space, is [2,3]

$$\frac{dy}{dx} = -\frac{(1 + y^2)x}{y}. \tag{5}$$

The two nullclines, i.e., curves along which the slope of the trajectories are either zero, $y_0(x)$, or unbounded, $y_\infty(x)$, are given by [2,3]

$$\frac{dy}{dx} = 0: \quad \text{along the curve } y_0(x), \text{ which is the } y\text{-axis}, \tag{6a}$$

$$\frac{dy}{dx} = \infty: \quad \text{along the curve } y_\infty(x), \text{ which is the } x\text{-axis}. \tag{6b}$$

These two nullclines divide the (x, y) phase plane into four domains which coincide with the standard four quadrants of the Cartesian plane. It also follows, from inspection of Eq. (5), that this equation is invariant under the three coordinate transformations:

$$T_1: \ x \to -x, \quad y \to y, \tag{7a}$$

$$T_2: \ x \to x, \quad y \to -y, \tag{7b}$$

$$T_3: \ x \to -x, \quad y \to -y. \tag{7c}$$

Note, however that the topological structure of the (x, y) phase plane for the trajectories, $y = y(x)$, for Eq. (1), is exactly that of the linear harmonic oscillator equations [2,3]

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x. \tag{8}$$

Since all the trajectories for the system given by Eq. (8) are closed curves, it follows that all the trajectories for Eq. (1) are also closed curves in the (x, y) phase space. Since closed curves in phase space correspond to periodic solutions [2,3], it follows that all the solutions to the original equation (1) are periodic. This specific result for Eq. (1) is clearly consistent with the more general conclusions presented in Ref. [4].

In summary, it has been shown, using phase-space methods, that Eq. (1) has periodic solutions for all initial conditions in phase space. A further consequence is that the fixed point, located at

$(\bar{x}, \bar{y}) = (0, 0)$, is a center [2,3], i.e., it has neutral stability. From these results, it can be concluded that the solutions to Eq. (1) do not contain singularities in its angular frequency, $\omega(A)$, as a function of the initial amplitude, A. The observed singularities occurring in the various methods to calculate approximate solutions for Eq. (1) are therefore artifacts of the perturbation methods and thus indicate limitations on these techniques. This study is consistent with the result that perturbation techniques generally have validity only for small oscillations about a given fixed-point or equilibrium solution.

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